ISRAEL JOURNAL OF MATHEMATICS, Vol. 21, Nos. 2-3, 1975

EVERY TRANSFORMATION IS BILATERALLY DETERMINISTIC

BY

DONALD S. ORNSTEIN AND BENJAMIN WEISS[†]

ABSTRACT

Every ergodic transformation T with a finite generator α, has another finite generator β, which refines α, and is bilaterally deterministic, i.e. $\bigvee_{|i|>n} T^i\beta$ is the full σ-algebra for every n.

1. Recall that a finite state stationary stochastic process $\{x_n\}_{-\infty}^{\infty}$ is said to be *deterministic* if x_0 is measurable with respect to the past – i.e., $\mathcal{B}(\cdots x_{-2}, x_{-1})$, the σ-field generated by $\{x_i\}_{i<0}$. By the stationarity, this is equivalent to the requirement that for all n, $\mathcal{B}(\cdots, x_{-n-1}, x_{-n}) = \mathcal{B}(\cdots, x_{-1}, x_0, x_1, \cdots)$. As a natural generalization we propose to call a process *bilaterally deterministic* if for all n $\mathcal{B}(\{x_i; |i| \geqq n\}) = \mathcal{B}(\{x_i; \text{all } i\})$. This is tantamount to requiring that a.s. given the far past $\{x_i\}_{i<-n}$, and the distant future, $\{x_i\}_{i>n}$, the present can be reconstructed. Naturally any deterministic process is bilaterally deterministic, but as we shall soon see the converse is far from being true. The determinism of a process is a property of its isomorphism class in the following sense. A stationary stochastic process $\{x_n\}$ defines a measure μ on $X = S^{Z}$, where S is the finite set of possible values of x_i. Let $\mathcal{B}$ be the product σ-field and $T: X \to X$ the coordinate shift, then $(X, \mathcal{B}, \mu, T)$ reflects the probabilistic structure of the x_n-process, and in fact the partition α that corresponds to the possible values of x_0 is a generator for T in the sense that $\bigvee_{-\infty}^{\infty} T^n \alpha = \mathcal{B}$. We say that two processes are isomorphic if the measure preserving transformations that they define are isomorphic as measure preserving transformations.

As is known, a f.s.s.s. process is deterministic if and only if the corresponding transformation has zero entropy, and thus if two f.s.s.s. processes are isomorphic, one is deterministic if and only if the other one is. In contrast it turns out that the property of bilateral determinism is not a property of the

[†] This research was supported in part by the National Science Foundation Research Grant NSF GP 33581X.

isomorphism class of a process. In fact our main result here is that any ergodic f.s.s.s. process is isomorphic to an ergodic f.s.s.s. process which is bilaterally deterministic. We devote the next section to proving this, and make some further comments concerning the interpretation in the final section. Similar results have been obtained independently by Gurevich and Furstenberg.

2. We take as data for describing an ergodic f.s.s.s. process an ergodic measure preserving transformation $(X, \mathcal{B}, \mu, T)$ and a finite generator $\alpha = \{A_0, \cdots, A_{k-1}\}$. We wish to find another *finite* generator β that defines a bilaterally deterministic process. The fact that both α and β are generators means of course that the stochastic processes that they define are isomorphic. The construction of β will be carried out in a sequence of steps. Let $\{n_i\}$ be a sequence tending rapidly to infinity and ε_i a sequence tending rapidly to zero—the precise rates required will become clear during the construction. We will guarantee that β be a generator by having it refine α, at every stage of the construction, and hence also in the limit.

Step 1. Build a Kakutani-Rohlin tower in the base E_1, height n_1^2, such that

$$\mu(X \sim \bigcup_0^{n_1^2-1} T^i E_1) < \varepsilon_1.$$

Think of the tower as being divided into n_1 groups each consisting of n_1 successive levels. The sets of β_1 will be: $B_i^{(1)}$, $0 \leqq i < k$; $B_{i,b}^{(1)}$, $B_{i,t}^{(1)}$, $0 \leqq i < k$; $B_{i,j}^{(1)}$, $0 \leqq i, j < k$. (The i coordinate will just give α.)

(i) Any point x in the first n_1 levels of the tower will be assigned to $B_{i,b}^{(1)}$ (the i is the one for which $x \in A_i$.)

(ii) Any point x in the last n_1 levels of the tower will be assigned to $B_{j,t}^{(1)}$ (the j is the one for which $x \in A_j$.)

(iii) Let

$$(i_{n_1+1}, i_{n_1+2}, \cdots, i_{2n_1}, i_{2n_1+1}, \cdots, i_{3n_1}, \cdots, i_{n_1(n_1-1)})$$

be the α-name of a point x in the n_1+1-level of the tower, that is

$$x \in A_{i_{n_1+1}}, Tx \in A_{i_{n_1+2}}, \cdots, T^j x \in A_{n_1+j+1}, \cdots.$$

Then x is assigned to the set

$$B_{i_{n_1+1},j}^{(1)} \text{ where } j = \sum_{l=1}^{n_1-2} i_{l \cdot n_1+1} \pmod k,$$

Tx is assigned to the set

$$B^{(1)}_{i_{n_1+2},j} \text{ where } j = \sum_{l=1}^{n_1-2} i_{l\cdot n_1+2}, \cdots$$

and so on to $T^{n_1-1}x$ which is assigned to the set

$$B^{(1)}_{i_{2n_1},j} \text{ where } j = \sum_{l=2}^{n_1-1} i_{l\cdot n_1} \pmod{k}.$$

(iv) Points x on all other levels, or not on the tower, are assigned to $B^{(1)}_j$ where the j is the one for which $x \in A_j$.

This stage defined a partition which we denote by β_1. Clearly β_1 is a refinement of α, and has the following features. On a large part of the space the β_1-name of a point enables one to identify the column level and correct n_1 successive mistakes in the α-name. To be more precise, if

$$x \in \bigcup_{j=3n_1+1}^{n_1^2-2n_1} T^j E_1,$$

and if for all $|i| \geqq n_1/2$ it is known to what set in β_1, $T^i x$ belongs; then using this information one can determine to what set in α, $T^i x$ belongs for *all i*. To do this use the bottom b-block and top t-block to position the correct level of x in the tower, and then the definition in part (iii) enables one to fill in missing α-symbols. Note that the set of exceptional points has at this point measure less than $5/n_1 + \varepsilon_1$.

At the ith stage we repeat the construction of step 1 (i)–(iii), getting a tower with base E_i with n_i and ε_i replacing n_1, ε_1. In part (iv) we replace α by β_{i-1}, that is to say the β_i constructed is the same as β_{i-1}, except for the modifications decreed by (i)–(iii). Now the sum total of the changes made to β_i in the succeeding steps is bounded by

$$c_i = \sum_{j>i} 3/n_j,$$

and our first requirement on the n_j is that Σc_i should converge. Furthermore, we shall require that

$$\sum_{1}^{\infty} (n_i c_i + \varepsilon_i)$$

converge. In that way, if G_i is defined to be the set of

$$x \in \bigcup_{j=3n_i+1}^{n_i^2-2n_i} T^j E_i$$

for which the β_i-name and the β_j-name throughout the n^2-levels of the tower remain the same for all $j > i$, we have that

$$\sum_{i=1}^{\infty} \mu(X \sim G_i) < +\infty.$$

From the foregoing it is clear that the β_i converge to a partition β which refines α. Furthermore, $\beta = \beta_i$ on the set G_i. By the Borel-Cantelli lemma we have a set of μ-measure 1, G and for $x \in G$ there is some index n such that $x \in G_i$ for all $i > n$. Suppose now that $x \in G$, and that for all $|i| \geqq m$ we are told to what set in β, $T^i x$ belongs. Then successively for all indices j such that $n_j > 2m$ we attempt to fill in the α-name, working under the hypothesis that $x \in G_j$. Eventually we succeed, since $x \in G_j$ for all large j. This clearly implies that

$$\bigvee_{-k}^{k} T^i\alpha \subset \bigvee_{|i|>l} T^i\beta$$

for all k, l, and since α is a generator one deduces that the β-process is bilaterally deterministic. We have therefore proved the following theorem:

THEOREM. *Given any ergodic finite state stationary stochastic process, there is an isomorphic ergodic finite state stationary stochastic process which is bilaterally deterministic.*

3. If one drops the requirement that the processes be finite state, the problems become less interesting since very naive codings could then make anything even deterministic — let alone bilaterally deterministic. The hypothesis of ergodicity was only made to simplify the presentation and is not really essential. All that we used was the existence of arbitrary Kakutani-Rohlin towers which exist for aperiodic transformations. On the periodic part any process is certainly bilaterally deterministic, so that the theorem holds in general.

One can interpret the theorem in another way. Instead of viewing some finite portion of the β-name as unknown, we can imagine the full β-name being transmitted over a noisy channel and being received with a finite number of errors. The theorem then states that allowing infinite codes (i.e., general measure theoretic isomorphism) one can encode perfectly, through a noisy channel which makes finite errors. Naturally both the encoding and decoding presuppose the entire signal, so this is not yet a "practical" result in any sense of the term.

Taking a closer look at the construction, the reader should be able to convince himself that perfect decoding is also possible if finitely many bits are dropped out or inserted in additional places. The fact that the true spacing between the $n_i - b$'s and $n_i \cdots t$'s signalling the bottom and top of the tower is known, means that the signal can be properly aligned.

The theorem applies of course to any Bernoulli-shift and says there that the triviality of the bilateral tail for the Bernoulli-shift is an accident due to the choice of independent generator. Since every process isomorphic to a B-shift is known to be very weak Bernoulli, we have the phenomenon of a kind of asymptotic independence (VWB) together with some sort of determinism — all of which goes to show that naive intuition can easily lead one astray. On the other hand, it is not hard to see that a weak Bernoulli process has a trivial bilateral tail, so that the construction given here serves also as an example of a VWB process that is not WB.

STANFORD UNIVERSITY
STANFORD, CALIFORNIA, U.S.A.

THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL